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GEOMETRICAL IMPLICATIONS OF THE EXISTENCE OF VERY SMOOTH BUMP FUNCTIONS IN BANACH SPACES

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ABSTRACT

We show that if X is a Banach space and if there is a non-zero real-valued C^∞-smooth function on X with bounded support, then either X contains an isomorphic copy of $c_0(\mathrm{N})$, or there is an integer k greater than or equal to 1 such that X is of exact cotype $2k$ and, in this case, X contains an isomorphic copy of $l^{2k}(\mathrm{N})$. We also show that if X is a Banach space such that there is on X a non-zero real-valued C^4-smooth function with bounded support and if X is of cotype q for $q < 4$, then X is isomorphic to a Hilbert space.

1. Introduction

Throughout this paper, a bump function on a Banach space X is a non-zero real-valued function on X with bounded support. We shall say that X is C^k-smooth if there exists a C^k-smooth bump function on X, and the set of such functions will be denoted $C^k(X)$. A norm on X is said (shortly) to be C^k-smooth if it is C^k-smooth away from the origin. If there exists on X an equivalent C^k-smooth norm, then $C^k(X)$ is not empty, the converse being open.

We are concerned with the geometrical implications of the existence of a C^k-smooth bump function on X where k is an integer greater than or equal to 2 or $k = +\infty$.

Let us first recall some definitions and results.

c_0 (or $c_0(\mathrm{N})$) denotes the Banach space of all sequences (x_n) of real numbers satisfying $\lim_{n\to\infty} x_n = 0$. We shall need the following result due to Bessaga and

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Pelczynski [1]: A Banach space does not contain a subspace isomorphic to c_0 if and only if for every sequence (x_i) in X such that $(\Sigma_{i=1}^n \varepsilon_i x_i)$ is bounded for every choice of signs, then $(\Sigma_{i=1}^n \varepsilon_i x_i)$ is convergent for every choice of signs, and the set $K = \bigcup_{i=1}^\infty K_n$ is relatively compact in X, where $K_n = \{\Sigma_{i=1}^n \varepsilon_i x_i : \varepsilon_i = 1 \text{ or } -1 \text{ for } i = 1, \ldots, n\}$.

Let us denote $\mathscr{S}_n = \{-1, +1\}^n$, X is said to be of cotype $q \in [2, +\infty]$ if and only if there is a constant C such that, for all finite subsets $(x_1, \ldots, x_n)$ of X,

$$\left(\sum_{i=1}^n \| x_i \|^q\right)^{1/q} \leqq C(1/2^n) \sum_{\varepsilon \in \mathscr{S}_n} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\| \qquad \text{if } q \text{ is finite,}$$

$$\sup_{1 \leqq i \leqq n} \| x_i \| \leqq C(1/2^n) \sum_{\varepsilon \in \mathscr{S}_n} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\| \qquad \text{if } q = +\infty.$$

Analogously, we say that X is of type p if there is a constant C such that, for all finite subsets $(x_1, \ldots, x_n)$ of X,

$$(1/2^n) \sum_{\varepsilon \in \mathscr{S}_n} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\| \leqq C\left(\sum_{i=1}^n \| x_i \|^p\right)^{1/p}.$$

In the definitions above, one can replace, using Kahane inequalities, $(1/2^n) \Sigma_{\varepsilon \in \mathscr{S}_n} \| \Sigma_{i=1}^n \varepsilon_i x_i \|$ by $((1/2^n) \Sigma_{\varepsilon \in \mathscr{S}_n} \| \Sigma_{i=1}^n \varepsilon_i x_i \|^2)^{1/2}$. Every Banach space X is of type 1 and of cotype ∞. If X is of type p then X is of type $p' < p$, and if X is of cotype q then X is of cotype $q' > q$. We shall need the following characterization of Hilbert spaces due to Kwapien [12]: a Banach space X is isomorphic to a Hilbert space if and only if it is of type 2 and of cotype 2. We say that X is of exact type p if X is of type p and X is not of type p' for $p' > p$, and X is of exact cotype q if X is of cotype q and X is not of cotype q' for $q' < q$. Let us mention that $l^p(\mathrm{N})$ is of exact type $\inf(p, 2)$ and of exact cotype $\sup(p, 2)$ for $1 \leqq p < +\infty$ and that $c_0(\mathrm{N})$ has only type 1 and cotype $+\infty$ and no more. We refer the reader to [8] and [17] for further details on these notions.

We now recall some basic facts about the existence of smooth bump functions on Banach spaces. First it is elementary to check that, for every measured space (Ω, A, μ) and every even integer $p \geqq 2$, the norm in $L^p(\Omega, A, \mu)$ is C^∞-smooth. Kuiper (see [19]) showed that there is on c_0 an equivalent C^∞-smooth norm. In fact there is on c_0 an equivalent norm analytic away from the origin [8]. On the other hand, Kurzweil [11] showed that:

(1) $\mathscr{C}[0, 1]$, the space of continuous real functions on $[0, 1]$, is not C^1-smooth.

(2) $l^p(\mathbf{N})$ is not C^∞-smooth whenever p is not an even integer.

Leach and Whitfield [13] generalized (1) by showing that if the norm of X is rough, equivalently, if the dual unit ball $B(X^*)$ does not admit slices determined by elements of X of arbitrarily small diameter, then X is not C^1-smooth. Actually, if X is C^1-smooth, then X is an Asplund space (see [3]) the converse being true if X is separable and open in general. Thus, at least for separable spaces, C^1-smooth spaces are characterized. Further investigations on C^1-smooth spaces can be found in [21] and a thorough exposition will appear in a forthcoming book of G. Godefroy and V. E. Zizler [9].

On the other hand, the geometry of Banach spaces on which there exists a bump function with higher properties of smoothness is far from being elucidated, although some significant progress has already been made in this direction. We have seen that c_0 is C^∞-smooth. Suppose now that X is a Banach space containing no isomorphic copy of c_0. If one can find on X a C^1-smooth bump function f such that f' is locally uniformly continuous, then X is superreflexive [6], [7]. If one can find on X a C^1-smooth bump function f with locally Lipschitzian derivative (this is the case if X is C^2-smooth), then X is superreflexive and of type 2 [7]. The relationships between the existence of a C^2-smooth bump function on X and the geometry of X has been further investigated in [5], [15], [16] and [18].

Let us outline the content of this paper. One of the tools that we shall use is the Taylor formula: for $f \in C^{k+1}(X)$, $x \in X$ and $h \in X$, we have

$$\begin{aligned} f(x+h) - f(x) &\\ = f'(x)(h) + \tfrac{1}{2}f''(x)(h,h) + \cdots + (1/k!)f^{(k)}(x)(h,\ldots,h) + R_k(x,h) &\\ = P_k(x)(h) + R_k(x,h) & \end{aligned}$$

with $|R_k(x,h)| \leqq C(x) \cdot \| h \|^{k+1}$.

In order to estimate $P_k(x)(h)$, we are led to study the k-linear functionals. This is done in Section 2.

We then prove in Section 3 the following abstract version of the result (2) of Kurzweil: if X is a C^∞-smooth Banach space and if $X \not\supset c_0$, then X is of exact cotype p for some even integer $p \geqq 2$. We shall also obtain in this section that a C^4-smooth Banach space of cotype $q < 4$ is necessarily isomorphic to a Hilbert space.

In Section 4, we show that if there exists on X a C^p-smooth bump function f (p integer, $p \geqq 2$) and if every subspace of X contains a subspace of cotype p, then X is of cotype p. As a consequence we get a result of B. M. Markharov

[14]: if there exists on X a C^2-smooth bump function f and if every (infinite dimensional) subspace of X contains an infinite dimensional subspace isomorphic to a Hilbert space, then X is isomorphic to a Hilbert space. Actually, our argument is a generalisation and a simplification of Makharov's proof.

We then apply in Section 5 the result of Section 4 to show that if X is of exact cotype p (p integer, $p \geqq 2$) and if X is C^p-smooth, then there is an infinite dimensional subspace Y of X and a p-linear (symetric) continuous form Φ on Y such that, for every $y \in Y$, $\Phi(y, y, \ldots, y) \geqq \| y \|^p$. Finally, using a diagonalizing argument, we exhibit a subspace Z of Y which is isomorphic to $l^p(\mathbf{N})$. As a consequence of our previous results, we get that if X is a C^∞-smooth Banach space, then either X contains an isomorphic copy of $c_0(\mathbf{N})$ or there is an even integer $p \geqq 2$ such that X contains an isomorphic copy of $l^p(\mathbf{N})$, thus answering a question of V. E. Zizler.

Some of the techniques used here have been introduced by Kurzweil in [11] and developed in [2], [5], [7], [15], [16], and [18].

2. k-Linear forms on a Banach space

Throughout this section, X denotes an infinite dimensional Banach space. Let k be an integer greater than 1, and Q be a k-linear continuous form on X. We recall that

$$\| Q \| = \sup\{ | Q(x_1, x_2, \ldots, x_k) | : \| x_1 \| \leqq 1, \ldots, \| x_k \| \leqq 1 \} < +\infty$$

and we shall write shortly $Q(x)$ in place of $Q(x, \ldots, x)$ for $x \in X$, when no confusion is possible. Let $k_1, \ldots, k_m$ be integers greater than 1 and let Q_i be a k_i-linear continuous functional on X, for $1 \leqq i \leqq m$. Our aim in this section is to find conditions on X such that for any $\varepsilon > 0$, there exists an $x \in X$, $x \neq 0$, such that, for $1 \leqq i \leqq m$,

$$| Q_i(x) | \leqq \varepsilon \| x \|^{k_i}.$$

We first observe that the situation is simple whenever all the k_i are odd:

Lemma 2.1. *Let R_i be a k_i-linear continuous form on X with k_i odd integers for $1 \leqq i \leqq m$. Then, for every $(m+1)$-dimensional subspace H of X, for every $\delta > 0$, there exists $x \in H$, $\| x \| = \delta$, satisfying, for all $i \in \{1, \ldots, m\}$, $R_i(x) = 0$.*

Proof. Let H be an $(m+1)$-dimensional subspace of X and $\delta > 0$ be fixed. Denote by R the application from the sphere of H centered at 0 of radius

δ into $\mathbf{R}^m$ given by $R(x)=(R_1(x),R_2(x),\dots,R_m(x))$. R is continuous and odd. According to the Borzuk–Ulam antipodal theorem (see e.g. [4] Cor. 4.2), there is an $x\in H$, $\|x\|=\delta$, such that $R(x)=0$.

We now study the case with k_i even numbers. In the following two lemmas, we use the approach of [18] Lemma 1.4:

Lemma 2.2. *Let k be an integer, $k\geqq 2$, Q be a k-linear continuous form on X, and $x_1,x_2,\dots,x_n$ be elements of X. Then there is a constant C depending on k such that*

$$\left|\sum_{i=1}^n Q(x_i)\right|\leqq C\,\|Q\|\,\left\|\sum_{i=1}^n \varepsilon_i x_i\right\|_2^k$$

where $(\varepsilon_i)_{i=1,\dots,n}$ are independent Bernoulli variables and

$$\left\|\sum_{i=1}^n \varepsilon_i x_i\right\|_2=\left\|\sum_{i=1}^n \varepsilon_i x_i\right\|_{L^2(\Omega,A,\mu)}=\left((1/2^n)\sum_{\varepsilon}\left\|\sum_{i=1}^n \varepsilon_i x_i\right\|^2\right)^{1/2}$$

where the summation over ε is taken over all choice of signs $\varepsilon=(\varepsilon_i)_{i=1}^n$.

Proof. Let $(\varepsilon_i^j)_{1\leqq i\leqq n,1\leqq j\leqq k-1}$ be independent Bernoulli variables on a probability space (Ω,A,μ) (therefore $\mu(\{x\in\Omega:\varepsilon_i^j(x)=1\})=\frac{1}{2}$ and $\mu(\{x\in\Omega:\varepsilon_i^j(x)=-1\})=\frac{1}{2}$).

We have

$$\sum_{i=1}^n Q(x_i,x_i,\dots,x_i)$$

$$=\int_\Omega\sum_{i=1}^n Q\left(x_i,x_i,\dots,\varepsilon_i^1x_i,\sum_{j=1}^n\varepsilon_j^1x_j\right)d\mu$$

$$=\int_\Omega\sum_{i=1}^n Q\left(x_i,x_i,\dots,\varepsilon_i^2x_i,\sum_{j=1}^n\varepsilon_j^2\varepsilon_j^1x_i,\sum_{j=1}^n\varepsilon_j^1x_j\right)d\mu$$

$$=\int_\Omega Q\left(\sum_{j=1}^n\varepsilon_j^{k-1}x_j,\sum_{j=1}^n\varepsilon_j^{k-1}\varepsilon_j^{k-2}x_j,\dots,\sum_{j=1}^n\varepsilon_j^1\varepsilon_j^2x_j,\sum_{j=1}^n\varepsilon_j^1x_i\right)d\mu,$$

therefore

$$\left|\sum_{i=1}^{n} Q(x_i)\right|$$
$$\leqq \| Q \| \int_{\Omega} \left\| \sum_{j=1}^{n} \varepsilon_j^{k-1} x_j \right\| \left\| \sum_{j=1}^{n} \varepsilon_j^{k-1}\varepsilon_j^{k-2} x_i \right\| \cdots \left\| \sum_{j=1}^{n} \varepsilon_j^{1} x_j \right\| d\mu;$$

using Jensen's inequality and the independence of the ε_i^j's we obtain

$$\left|\sum_{i=1}^{n} Q(x_i)\right|$$
$$\leqq \| Q \| \left\| \sum_{j=1}^{n} \varepsilon_j^{k-1} x_j \right\|_k \left\| \sum_{j=1}^{n} \varepsilon_j^{k-1}\varepsilon_j^{k-2} x_j \right\|_k \cdots \left\| \sum_{j=1}^{n} \varepsilon_j^{1} x_j \right\|_k$$

so, using Kahane's inequalities, there is a constant C depending on p such that

$$\left|\sum_{i=1}^{n} Q(x_i)\right| \leqq C \| Q \| \left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\|_2^k .$$

LEMMA 2.3. *Let Q_i be a k_i-linear continuous form on X with $k_i \geqq 2$ for $1 \leqq i \leqq m$, and let $\varepsilon > 0$. We assume that for every $x \in X$, $\| x \| = 1$,*

$$\varepsilon \leqq \sup\{ |Q_j(x)| : 1 \leqq j \leqq m \}.$$

Then X is of cotype k, where $k = \sup\{k_i : 1 \leqq i \leqq m\}$.

PROOF. Let $x_1, x_2, \ldots, x_n$ be elements of X. Since $\| \Sigma_{i=1}^n \varepsilon_i x_i \|_2 \geqq \| x_i \|$ for every $i \in \{1, \ldots, n\}$, replacing x_i by λx_i we may assume that, for every $i \in \{1, \ldots, n\}$, $\| x_i \| \leqq 1$ and $\| \Sigma_{i=1}^n \varepsilon_i x_i \|_2 \geqq 1$. We then have

$$\begin{aligned}\varepsilon\left(\sum_{i=1}^{n} \| x_i \|^k\right) &\leqq \sum_{i=1}^{n}\left(\sum_{j=1}^{m} |Q_j(x_i)|\right)\\ &\leqq m \max \left|\sum_{i=1}^{n} |Q_j(x_i)\right|\\ &\leqq m \sum_{i=1}^{n} |Q_{j_0}(x_i)|\end{aligned}$$

for some $j_0 \in \{1, \ldots, m\}$. Without loss of generality, we may assume $j_0 = 1$. Let

$$I^+ = \{i \in \{1, \ldots, n\} : Q_1(x_i) > 0\} \quad \text{and} \quad I^- = \{i \in \{1, \ldots, n\} : Q_1(x_i) \leqq 0\}.$$

We can assume further that

$$\sum_{i\in I^+} Q_1(x_i) \geqq -\sum_{i\in I^-} Q_1(x_i).$$

Under this condition, we obtain

$$\begin{aligned}\varepsilon\left(\sum_{i=1}^{n} \| x_i \|^k\right) &\leqq 2m \sum_{i\in I^+} Q_1(x_i)\\ &\leqq 2m \| Q_1 \| \left\| \sum_{i\in I^+} \varepsilon_i x_i \right\|_2^{k_1}\\ &\leqq 2m \| Q_1 \| \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|_2^{k_1}\\ &\leqq 2m \| Q_1 \| \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|_2^{k}.\end{aligned}$$

Therefore X is of cotype k.

Proposition 2.4. *Let Q_i be a k_i-linear continuous form on X with $1 \leqq k_i \leqq k$ for $1 \leqq i \leqq m$, and assume X is not of cotype k.*

Then, for every $\varepsilon > 0$ there is an infinite dimensional subspace Y of X such that

$$\forall x \in Y \quad \forall i \in \{1, \ldots, n\} \qquad |Q_i(x)| \leqq \varepsilon \| x \|^{k_i}.$$

Proof. If some of the Q_j's are 1-linear, say $Q_1, Q_2, \ldots, Q_p$, replacing X by $\bigcap_{j=1}^{p} \operatorname{Ker} Q_j$, which is a finite codimensional subspace of X and therefore is not of cotype k, it is enough to consider only the Q_j's which are k_j-linear with $k_j \geqq 2$. Therefore we may assume that $k_j \geqq 2$ for $1 \leqq i \leqq m$.

It is enough to construct an increasing sequence of subspaces Y_n of X such that $\dim Y_n = n$, and a sequence of real numbers (ε_n), $0 < \varepsilon_n < \varepsilon$, satisfying

H_n: for every $x \in Y_n$, $\| x \| = 1$, and for every $i \in \{1, \ldots, m\}$, $|Q_i(x)| \leqq \varepsilon - \varepsilon_n$.

According to Lemma 2.3, H_1 is satisfied with $\varepsilon_1 = \varepsilon/2$. Let us assume that H_n is satisfied and let us construct Y_{n+1} such that:

For every $x \in Y_{n+1}$, $\| x \| = 1$, and for every $i \in \{1, \ldots, m\}$, $|Q_i(x)| \leqq \varepsilon - \varepsilon_n/2$, and this will prove the lemma.

Let $\delta_n > 0$ be specified later, A_n be a δ_n-net of the unit ball of Y_n and Z_n be a finite codimensional subspace of X satisfying, for every $x \in Y_n$ and for every $z \in Z_n$,

$$\| x \| \leqq 2 \| x + z \|. \tag{1}$$

Now, applying Lemma 2.3 to the space Z_n which is not of cotype k, we can choose $z_n \in Z_n$, $\| z_n \| = 1$, such that for every $i \in \{1, \dots, m\}$, for every $j \in \{1, \dots, k_i\}$ and for every $y \in A_n$,

$$(2) \qquad | Q_i^j(y)(z_n) | \leqq \delta_n,$$

where we have written $Q_i(y+z) = Q_i(y) + Q_i^1(y)(z) + \cdots + Q_i^{k_i}(y)(z)$ with $Q_i^j(y)$ j-linear continuous form on X (note that $Q_i^{k_i}(y)(z) = Q_i(z)$).

Let us consider Y_{n+1} the subspace spanned by Y_n and z_n. Y_{n+1} is a $(n+1)$-dimensional subspace of X and we claim that H_{n+1} is satisfied if δ_n is small enough. Indeed, let $t \in Y_{n+1}$ such that $\| t \| \leqq 1$. We have $t = x + \lambda z_n$ with $x \in Y_n$. It follows from (1) that $\| x \| \leqq 2$ and $| \lambda | \leqq 3$. Let us choose $y \in A_n$ such that $\| x - y \| \leqq 2\delta_n$. We have then, for every $i \in \{1, \dots, m\}$,

$$(3) \qquad | Q_i(x + \lambda z_n) - Q_i(y + \lambda z_n) | \leqq \| Q_i \| \alpha(\delta_n) \quad \text{with } \lim_{\delta_n \to 0} \alpha(\delta_n) = 0$$

and, on the other hand,

$$(4) \qquad \begin{aligned} | Q_i(y + \lambda z_n) | &\leqq | Q_i(y) | + | Q_i^1(y)(\lambda z_n) | + \cdots + | Q_i^{k_i}(y)(\lambda z_n) | \\ &\leqq (\varepsilon - \varepsilon_n) + 3\delta_n + \cdots + 3^{k_i}\delta_n. \end{aligned}$$

It follows from (3) and (4) that, for every $i \in \{1, \dots, m\}$, $| Q_i(x + \lambda z_n) | \leqq \varepsilon - \varepsilon_n/2$ whenever δ_n is chosen small enough (the choice of δ_n depends only on ε_n and on the $\| Q_i \|$'s).

We shall also need the following slightly stronger version of Proposition 2.4:

Proposition 2.5. *Let Q_i be a k-linear continuous form on X for $1 \leqq i \leqq m_0$, and let Q_i be a k_i-linear continuous form on X with $1 \leqq k_i \leqq k - 1$ for $m_0 + 1 \leqq i \leqq m$.*

Assume that:

(1) *X is not of cotype $k - 1$,*

(2) *for every $i \in \{1, \dots, m_0\}$, for every $\varepsilon > 0$ and for every infinite dimensional subspace Y of X, there is an $x \in Y$, $x \neq 0$, such that $| Q_i(x) | \leqq \varepsilon \| x \|^k$.*

Then, for every $\varepsilon > 0$, there is an infinite dimensional subspace Y of X such that, for every $i \in \{1, \dots, m\}$ and every $x \in Y$,

$$| Q_i(x) | \leqq \varepsilon \| x \|^{k_i}.$$

Proof. We fix $\varepsilon > 0$ and we proceed by induction on the number m_0 of k-linear continuous forms Q_i on X. By Proposition 2.4, Proposition 2.5 is satisfied if $m_0 = 0$. Assume now that Proposition 2.5 is true for $m_0 - 1$; therefore, under the hypothesis of Proposition 2.5, we can find an infinite dimensional subspace Y of X such that, for every $i \in \{1, \ldots, m\}$, $i \neq m_0$, and every $x \in Y$, $|Q_i(x)| \leqq \varepsilon \| x \|^{k_i}$. We can then construct an infinite dimensional subspace Z of Y (using (2) for $i = m_0$) in the same way as in the proof of Proposition 2.4 such that, for every $x \in Z$, $|Q_{m_0}(x)| \leqq \varepsilon \| x \|^k$, and this proves Proposition 2.5.

We now give a consequence of Lemma 2.1 and of Proposition 2.4. For this purpose, we need to introduce the following definition:

DEFINITION. An application P from X into $\mathbf{R}$ is called a polynomial if we can write, for every $x \in X$,

$$P(x) = P_1(x) + P_2(x, x) + \cdots + P_k(x, x, \ldots, x),$$

where P_j is a j-linear continuous form on X for $1 \leqq j \leqq k$. k is called the degree of P.

PROPOSITION 2.6. *Let $P_1, \ldots, P_m$ be polynomials of degree less than or equal to k on a Banach space X, and assume that X is not of cotype k if k is even and that X is not of cotype $k-1$ if k is odd. Let $\varepsilon > 0$ and $0 < a < 1$. Then there exists $x \in X$, $\| x \| = a$, such that for every $i \in \{1, \ldots, m\}$.*

$$|P_i(x)| < \varepsilon \quad \textit{and} \quad |P_i(-x)| < \varepsilon.$$

PROOF. We split the P_i's into multilinear continuous forms that we call Q_j, $1 \leqq j \leqq j_0$, and R_j, $1 \leqq i \leqq j_1$, with the degree of the Q_j's even and the degree of the R_j's odd. Quantities ε, a and the P_i's being fixed we can find, by Proposition 2.4, an infinite dimensional subspace H of X satisfying, for every $x \in H$, $\| x \| = a$, and every $j \in \{1, \ldots, j_n\}$,

$$|Q_j(x)| < \varepsilon / k. \tag{5}$$

Since $\dim H \geqq j_0 + 1$, using Lemma 2.1, there exists $x \in H$, $\| x \| = a$, such that for all $j \in \{1, \ldots, j_1\}$,

$$R_j(x) = 0. \tag{6}$$

We deduce from (5) and (6) the existence of $x \in X$ satisfying

$$\forall i \in \{1, \ldots, m\} \qquad |P_i(x)| < \varepsilon \quad \text{and} \quad |P_i(-x)| < \varepsilon$$

which proves the proposition.

3. Cotype of C^∞-smooth Banach spaces

In this section, we prove the following result:

THEOREM 3.1. *Assume that X is a C^∞-smooth Banach space which does not contain an isomorphic copy of c_0. Then there exists an integer $k \geqq 1$ such that X is of exact cotype $2k$.*

For the proof of Theorem 3.1, we need one more elementary lemma:

LEMMA 3.2. *Let K be a compact of X, $f \in C^{k+1}(X)$, $k \geqq 1$ and $\varepsilon > 0$. Then there exists $C > 0$ and $\delta > 0$ such that, for every $x \in K$ and for every $h \in X$, $\| h \| < \delta$,*

$$\| f(x+h) - f(x) - P_k(x)(h) \| \leqq C \| h \|^{k+1}.$$

PROOF. Since $f \in C^{k+1}(X)$, in particular $f \in C^k(X)$ and $f^{(k)}$ is locally Lipschitz on X. Therefore, there exists $\delta > 0$ and $M > 0$ such that, for every $x \in K$ and every $h \in X$, $\| h \| < \delta$,

$$\| f^{(k)}(x+h) - f^{(k)}(x) \| \leqq M \| h \|.$$

Now, using Taylor's formula, if $x \in K$ and $h \in X$, $\| h \| < \delta$,

$$\begin{aligned} &| f(x+h) - f(x) - P_k(x)(h) | \\ &= \left| \int_0^1 [f^{(k)}(x+th)(h) - f^{(k)}(x)(h)]\,(1-t)^{k-1}/(k-1)!\,dt \right| \\ &\leqq (1/k!) \| h \|^k \sup_{t \in [0,1]} \| f^{(k)}(x+th) - f^{(k)}(x) \| \\ &\leqq (M/k!) \| h \|^{k+1}. \end{aligned}$$

REMARK. Let a satisfy $a \leqq \delta$ and $a^{k+1-q} \leqq \varepsilon / C$, for some q such that $k < q < k+1$. We have then, for $x \in K$ and $h \in X$, $\| h \| \leqq a$,

$$| R_k(x,h) | = | f(x+h) - f(x) - P_k(x)(h) | \leqq \varepsilon \| h \|^q.$$

PROOF OF THEOREM 3.1. Let X be a Banach space not containing a subspace isomorphic to c_0, and let f be a C^∞-smooth bump function on X such that $f(0) = 1$ and $f(x) = 0$ if $\| x \| \geqq 1$. According to Fabian et al. [7], X has type 2 and so [15] X has also a finite cotype. Therefore

$$C(X) = \inf \{ q \in \mathbf{R} \colon X \text{ has cotype } q \} < +\infty.$$

Denote $k = [C(X)]$ the greatest integer less than or equal to $C(X)$ and $p = 2p'$

the greatest even integer less than or equal to $C(X)$. Assume that X is not of cotype p, and choose $q \in \mathbf{R}$ satisfying

$$C(X) < q < [C(X)] + 1.$$

Since X is of cotype q, there exists a constant C such that

$$\forall n \in \mathrm{N} \quad \forall x_1, x_2, \ldots, x_n \in X \qquad \sum_{i=1}^{n} \| x_i \|^q \leqq C \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|^q.$$

Fix ε such that $0 < \varepsilon < (C+1)^{-1}$ By induction, we construct vectors $x_n \in X$ satisfying

$$\text{(1)} \qquad x_0 = 0,$$

$$\text{(2)} \qquad K_{n-1} = \left\{ \sum_{i=1}^{n} \varepsilon_i x_i : \varepsilon_i = \pm 1, i = 1, \ldots, n \right\},$$

$$\text{(3)} \quad H_n = \{h \in X : \forall x \in K_{n-1} : P_k(x)(h) \leqq 3^{-n} \text{ and } P_k(x)(-h) \leqq 3^{-n}\}.$$

Using Proposition 2.6, for every $a \in (0, 1)$, $H_n \cap \{h \in X : \| h \| = a\}$ is not empty.

$$\text{(4)} \quad \begin{aligned} E_n = \{h \in H_n : \| h \| \leqq 1 \text{ and, for every } x \in K_{n-1} : |R_k(x, h)| \leqq \varepsilon \| h \|^q \text{ and} \\ |R_k(x, -h)| \leqq \varepsilon \| h \|^q\}. \end{aligned}$$

According to Lemma 3.2 and the remark following it, $E_n \cap \{h \in X : \| h \| = a\}$ is not empty whenever a is small enough.

$$\text{(5)} \qquad x_n \in E_n, \qquad \| x_n \| \geqq \tfrac{1}{2} \sup\{ \| h \| : h \in E_n\}.$$

Once this induction is completed, we set $K = \bigcup_{n \in \mathrm{N}} K_n$.

First case. K is not bounded

There exists an integer n such that:

$$\sup\{ \| x \| : x \in K_{n-1}\} \leqq 1 \quad \text{and} \quad \sup\{ \| x \| : x \in K_n\} > 1.$$

Without loss of generality, we can assume that $\| \Sigma_{i=1}^{n} x_i \| > 1$, and we obtain

$$\begin{aligned} 1 &= \left| f\left(\sum_{i=0}^{n} x_i \right) - f(0) \right| \\ &\leqq \sum_{i=1}^{n} \left| f\left(\sum_{j=0}^{i} x_j \right) - f\left(\sum_{j=1}^{i-1} x_j \right) \right| \end{aligned}$$

$$\leqq \sum_{i=1}^{n} (\varepsilon \| x_i \|^q + 3^{-i})$$

$$\leqq \varepsilon \left(\sum_{i=1}^{n-1} \| x_i \|^q + 1 \right) + \tfrac{1}{2} \leqq \varepsilon (C+1) + \tfrac{1}{2},$$

therefore this case is not possible.

Second case. K is bounded

Since X does not contain a subspace isomorphic to c_0, K is relatively compact and $\lim_{n\to\infty} \| x_n \| = 0$ [1]. Using Lemma 3.2 and the remark following it, there exists $\delta \in (0, 1)$ such that, for every $x \in K$ and every $h \in X$, $\| h \| \leqq \delta$,

$$| R_k(x, h) | \leqq \varepsilon \| h \|^q.$$

Therefore, since $K_{n-1} \subset K$, if $h \in H_n$ and $\| h \| \leqq \delta$, then $h \in E_n$.

Since for every $a \in (0, 1)$, $H_n \cap \{ h \in X : \| h \| = a \}$ is not empty, by (5), $\| x_n \| > \delta/2$ for every $n \in \mathbf{N}$, which contradicts the fact that $\lim_{n\to\infty} \| x_n \| = 0$ and that δ is independent of n.

A closer look at the proof of Theorem 3.1 shows us that we have actually proved more. Let us first introduce the following definition which appears in [18]:

Definition. Let $q > 1$ and denote k the greatest integer strictly less than q. Let f be a real-valued function on X. We say that f is H^q-smooth if f is C^k-smooth and if for every $x \in X$ there exists $\delta, M > 0$ such that $\| y - x \| < \delta$ and $\| z - x \| < \delta$ implies $| f^k(y) - f^k(z) | \leqq M \| y - z \|^{q-k}$. Note that if f is C^{k+1}-smooth, then f is H^q-smooth.

Modifying in an obvious way Lemma 3.2, one can show, using the same proof as the proof of Theorem 3.1:

Theorem 3.1 bis. *Suppose that X is a Banach space of cotype $q < +\infty$ and that X is not of cotype k, where k is the greatest even integer less than or equal to q. Then there does not exist on X an $H^{q'}$-smooth bump function for $q' > q$.*

In particular, using the fact due to Kwapien [12] that a Banach space of type 2 and cotype 2 is isomorphic to a Hilbert space, we get the following improvement of a result of [5] (Theorem 1(b)):

Corollary 3.3. *Assume that X is a Banach space of cotype q with $q < 4$ and that there is on X a C^4-smooth (or merely $H^{q'}$-smooth for $q' > q$) bump function. Then X is isomorphic to a Hilbert space.*

Of course, applying Theorem 3.1 to L^p spaces, we have

COROLLARY 3.4 (Kurzweil–Bonic–Frampton). *Let* (Ω, A, μ) *be a measured space such that* $L^p(\Omega, A, \mu)$ *(denoted shortly* L^p*) is infinite dimensional. Suppose that* $p \geqq 1$ *and that* p *is not an even integer. Then there does not exist on* L^p *a* C^∞*-smooth (or merely* $H^{p'}$*-smooth for* $p' > p$*) bump function.*

PROOF. Indeed, if $p < 2$, L^p is of type p but not of type 2, and, if $p \geqq 2$, L^p is of cotype p but not of cotype q for $q < p$ (see for instance [20]).

We conclude by giving an application on k-linear continuous forms on X, or, more generally, on polynomials on X:

COROLLARY 3.5. *Assume that* X *is a Banach space of finite cotype but is not of exact cotype* p *with* p *an even integer. Then, for every polynomial* Q *on* X *(whatever is its degree) and for every* $\varepsilon > 0$, *there exists* $x \in X$, $\| x \| = 1$, *such that* $|Q(x)| < \varepsilon$.

PROOF. Indeed, otherwise there would exist a polynomial Q on X and $\varepsilon > 0$ such that for every $x \in X$, $\| x \| = 1$, we have $Q(x) \geqq \varepsilon$. Let f be a C^∞-smooth real function on $\mathbf{R}$ such that $f(0) = 1$ and $f(t) = 0$ if $|t| \geqq \varepsilon$. The function $\varphi : X \to \mathbf{R}$ defined by

$$\varphi(x) = f \circ Q(x) \qquad \text{if } \| x \| \leqq 1$$

$$\varphi(x) = 0 \qquad \text{if } \| x \| > 1$$

is a C^∞-smooth bump function on X, which contradicts the hypothesis.

4. Cotype and subspaces of C^p-smooth Banach spaces

THEOREM 4.1. *Let* X *be a Banach space and* p *be an integer. Suppose that there is on* X *a* C^p*-smooth bump function* φ *and that every subspace* Y *of* X *(*Y *infinite dimensional) contains an infinite dimensional subspace of cotype* p. *Then* X *is of cotype* p.

PROOF. Let us assume that the hypotheses of Theorem 4.1 hold and that X is not of cotype p. We shall find a contradiction in the same way as in [15]. Let φ be a C^p-smooth bump function on X satisfying $\varphi(0) = 1$ and $\varphi(x) = 0$ if $\| x \| \geqq 1$.

By induction, we construct a sequence (x_n) of elements of X, finite dimensional subspaces X_n of X, finite subsets A_n of X and finite subsets B_n of X^* satisfying:

(1) $x_1 \in X$, $\| x_1 \| = 1$,

and, for $n \geqq 2$,

(2) $A_n = \{(\Sigma_{i=1}^{n-1} a_i x_i)/2^n : 1 \leqq a_1, a_2, \ldots, a_{n-1} \leqq 2^n\}$,

(3) $X_n = \text{span}\{x_i : i < n\}$,

(4) B_n is a finite subset of X^* such that the restrictions of the elements of B_n to X_n form a 2^{-n}-net of $B(X_n^*)$,

(5) $\| x_n \| = 1$ and $y(x_n) = 0$ for every $y \in B_n$,

(6) for every $x \in A_n$, for every $q \leqq p$ and for every finite sequence $1 \leqq n_1 \leqq n_2 \leqq \cdots \leqq n_q = n$, $\varphi^{(q)}(x)(x_{n_1}, x_{n_2}, \ldots, x_{n_q}) \leqq \varepsilon(n, p)$ where $\varepsilon(n, p)$ satisfies, for every $n \in \mathbf{N}$,

$$p \cdot p! \sum_{n \leqq s_1 \leqq s_2 \leqq \cdots \leqq s_p < +\infty} \varepsilon(s_p, p) \leqq 1/3^n.$$

This construction is a straightforward application of Proposition 2.4 at each step.

Once this construction is completed, set $F = \overline{\bigcup_{n \in \mathbf{N}} X_n}$ and $A = \bigcup_{n \in \mathbf{N}} A_n$. ($F$ is the norm closed subspace spanned by the x_n's and the norm closure of A contains the ball of Y centered at 0 of radius $\frac{1}{2}$ since, according to conditions (3), (4) and (5), (x_n) is a basis of F with basis constant 2.) Since F is an infinite dimensional subspace of X, by hypothesis, we can find an infinite dimensional subspace Z of F of cotype p: there is a constant $C > 0$ such that, for every $n \in \mathbf{N}$ and every finite sequence $(y_1, y_2, \ldots, y_n)$ of elements of Z,

$$(*) \qquad \sum_{i=1}^{n} \| y_i \|^p \leqq C \left\| \sum_{i=1}^{n} \varepsilon_i y_i \right\|_2^p .$$

In fact, replacing Z by a subspace of Z and using a standard perturbation argument, we can assume that $Z = \overline{\text{span}}\{u_k : k \in \mathbf{N}\}$, where (u_k) is a block basis of (x_n) such that $u_k \in A$ for every $k \in \mathbf{N}$. Finally, let us fix $0 < \varepsilon < (C+1)^{-1}/2$.

We now construct by induction a sequence (y_n) of elements of A in the following way: we choose $y_0 = 0$. If we have constructed $y_0, y_1, \ldots, y_{n-1}$, then we set

(7) $k_n = \inf\{k \geqq n : y_p \in A_k \text{ for every } p < n\}$,

(8) $Z_n = \text{span}\{u_k : k \geqq k_n\}$,

(9) $E_n = \{y \in A \cap Z_n : \| y \| \leqq \frac{1}{2}$ and for every choice of signs $(\varepsilon_i)_{i=1}^n$
$|R_p(\Sigma_{i=1}^{n-1} \varepsilon_i x_i, \varepsilon_n y)| \leqq \varepsilon \| y \|^p\}$,

(10) $\alpha_n = \sup\{ \| y \| : y \in E_n\} > 0$,

(11) choose $y_n \in E_n$ such that $\| y_n \| \geqq \alpha_n/2$.

Once the y_n's are constructed, let $Y_n = \{\Sigma_{i=1}^n \varepsilon_i y_i : \varepsilon_i \in \{-1, +1\}^n\}$ and $Y = \bigcup_{n\in\mathbf{N}} Y_n \subset Z$. We claim that

$$(**) \qquad \left| P_p\left(\sum_{i=1}^{n-1} \varepsilon_i y_i\right)(y_n)\right| \leqq 1/3^n$$

and

$$(***) \qquad \left| P_p\left(\sum_{i=1}^{n-1} \varepsilon_i y_i\right)(-y_n)\right| \leqq 1/3^n.$$

Indeed, set $x = \Sigma_{i=1}^{n-1} \varepsilon_i y_i$ and $y_n = \Sigma_{r=k_n}^{l_n} a_k u_k = \Sigma_{s=k_n}^{m_n} b_s x_s$. Note that $k_n \geqq n$ and, since (x_n) is a basic sequence with basis constant $\leqq 2$, we have, for $s = 1, \ldots, m$, $|b_s| \leqq 2\| y_n \| \leqq 1$. Therefore, for $q \in \{1, 2, \ldots, p\}$, we have

$$\begin{aligned} |\varphi^{(q)}(x)(y_n)| &\leqq \left| \sum_{k_n \leqq s_1, s_2, \ldots, s_q \leqq m_n} b_{s_1} b_{s_2} \cdots b_{s_q} \varphi^{(q)}(x)(x_{s_1}, x_{s_2}, \ldots, x_{s_q})\right| \\ &\leqq \sum_{k_n \leqq s_1, s_2, \ldots, s_q \leqq m_n} |\varphi^{(q)}(x)(x_{s_1}, x_{s_2}, \ldots, x_{s_q})| \\ &\leqq q! \sum_{k_n \leqq s_1, s_2, \ldots, s_q \leqq m_n} |\varphi^{(q)}(x)(x_{s_1}, x_{s_2}, \ldots, x_{s_q})| \\ &\leqq p! \sum_{n \leqq s_1 \leqq s_2 \leqq \cdots \leqq s_q < +\infty} \varepsilon(s_q, p) \\ &\leqq 1/(p \cdot 3^n) \end{aligned}$$

and this proves (**) and (***).

It follows from (**), (***) and (9) that

$$\left| f\left(\sum_{i=1}^{n-1} \varepsilon_i y_i + y_n\right) - f\left(\sum_{i=1}^{n-1} \varepsilon_i y_i\right)\right| \leqq \varepsilon \| y_n \|^p + 1/3^n$$

and

$$\left| f\left(\sum_{i=1}^{n-1} \varepsilon_i y_i - y_n\right) - f\left(\sum_{i=1}^{n-1} \varepsilon_i y_i\right)\right| \leqq \varepsilon \| y_n \|^p + 1/3^n;$$

two cases are possible.

First case. Y is not bounded

Repeating word by word the argument of the case "K is not bounded" in the proof of Theorem 2.1, we see that the hypothesis "Y is not bounded" contradicts the choice of ε (here the inequality (*) is used).

Second case. Y is bounded

Since X is hereditarily cotype p, X does not contain a subspace isomorphic to c_0, therefore, by a result of Bessaga and Pelczynski [1], Y is relatively compact and $\lim_{n\to+\infty} \| y_n \| = 0$. Therefore, if we set

$$E = \{ y \in A : \| y \| \leqq 1 \text{ and for every } z \in Y,$$
$$|R(z,y)| \leqq \varepsilon \| y \|^p \quad \text{and} \quad |R(z,-y)| \leqq \varepsilon \| y \|^p \},$$

by Lemma 3.2, E contains all the points of A of norm $\leqq \delta$ for some $\delta > 0$, and this implies by (2), (9), (10) and (11) that $\liminf_{n\to+\infty} \| y_n \| \geqq \delta/2$, which is a contradiction. Therefore, X is of cotype p and the theorem is proved.

We recall that if (X_n) is a sequence of (finite or infinite dimensional) Banach spaces, then

$$(\bigoplus X_n)_p = \left\{ (x_n) : x_n \in X_n \text{ and } \sum_{n=1}^{+\infty} \| X_n \|^p < +\infty \right\}$$

is a Banach space equipped with the norm $\| (x_n) \| = (\Sigma_{n=1}^{+\infty} \| x_n \|^p)^{1/p}$. Let us denote $C_n > 0$ the cotype p constant of X_n. The following result is an immediate consequence of Theorem 4.1:

COROLLARY 4.2. *Assume that there exists on* $(\bigoplus X_n)_p$ *a* C^p*-smooth bump function, where p is an even integer* $\geqq 2$. *Then* $\limsup_{n\to+\infty} C_n < +\infty$.

In particular, if an infinite number of the X_n*'s are not of cotype p, then* $(\bigoplus X_n)_p$ *is not* C^p*-smooth.*

For instance, it follows from Theorem 3.1 and Corollary 4.2 that if $1 \leqq p < q < +\infty$, then $(\bigoplus l^q(\mathbf{N}))_p$ is never C^∞-smooth.

The following corollary is the main result of [15]:

COROLLARY 4.3. *Assume that X is a* C^2*-smooth Banach space such that every infinite dimensional subspace of X contains an infinite dimensional subspace of cotype* 2, *then X is isomorphic to a Hilbert space.*

In particular, if X is a C^2*-smooth Banach space such that every infinite dimensional subspace of X contains an infinite dimensional Hilbert space, then X is isomorphic to a Hilbert space.*

PROOF. The assumptions of Corollary 4.3 imply, by Theorem 4.1, that X is of cotype 2. On the other hand, according to Fabian et al. [7], a C^2-smooth Banach space which does not contain an isomorphic copy of c_0 is of type 2. Therefore, X being of type 2 and of cotype 2 is isomorphic to a Hilbert space by a result of Kwapien [12].

COROLLARY 4.4. *Let $q<4$ and assume that X is a hereditarily cotype q Banach space (i.e. every infinite dimensional subspace Y of X contains an infinite dimensional subspace H of cotype q) and that X is C^4-smooth. Then X is isomorphic to a Hilbert space.*

PROOF. Corollary 4.4 follows from Corollary 3.3 and Corollary 4.3.

5. Subspaces of C^p-smooth Banach spaces

In what follows, let p be an even integer, $p \geqq 2$, and assume that X is a C^p-smooth Banach space of exact cotype p. Our goal is to show that X contains an isomorphic copy of $l^p(\mathbf{N})$. Our first step is the following:

PROPOSITION 5.1. *Let p be an even integer, $p \geqq 2$, and X be a C^p-smooth Banach space of exact cotype p. Then there is an infinite dimensional subspace Z of X and a p-linear continuous symmetric form Φ on X such that, for every $x \in Z$, $\Phi(x, x, \ldots, x) \geqq \| x \|^p$. Moreover, none of the infinite dimensional subspaces of Z is of cotype $p-1$.*

PROOF. We shall use the same argument as in the proof of Theorem 3.1. Replacing X by a suitable infinite dimensional subspace of X, we can assume, by Theorem 4.1, that none of the infinite dimensional subspaces of X is of cotype $p-1$. In order to get a contradiction, assume that, for every p-linear continuous symmetric form Φ on X, for every $\varepsilon>0$ and for every infinite dimensional subspace Z of X, there exists $x \in Z$, $x \neq 0$, such that $\Phi(x, x, \ldots, x)<\varepsilon \| x \|^p$.

By induction, we construct a sequence (x_n) of elements of X and infinite dimensional subspaces Z_n of X in the following way:

Choose $x_0=0$ and $Z_0=X$. If $x_0, x_2, \ldots, x_{n-1}, Z_0, Z_1, \ldots, Z_{n-1}$ have been constructed, choose, by Proposition 2.5, an infinite dimensional subspace $Z_n \subset Z_{n-1}$ such that for every choice of signs $(\varepsilon_i)_{i=1}^{n-1}$, for every $q \leqq p$ and every $x \in Z_n$,

$$\left| \varphi^{(q)}\left(\sum_{i=1}^{n-1} \varepsilon_i x_i\right)(x, x, \ldots, x)\right| \leqq 1/(p \cdot 3^n).$$

Therefore, we get

$$\left| P_p\left(\sum_{i=1}^{n-1} \varepsilon_i x_i\right)(x)\right| \leqq 1/3^n.$$

Let

$$E_n = \Big\{ x \in Z_n : \ \| x \| \leqq 1 \text{ and for every choice of signs } (\varepsilon_i)_{i=1}^n,$$

$$\Big| R_p \Big(\sum_{i=1}^{n-1} \varepsilon_i x_i , \varepsilon_n x_n \Big) \Big| \leqq \varepsilon \| x \|^p \Big\}$$

and

$$\alpha_n = \sup\{ \| x \| : x \in E_n \}.$$

Choose $x_n \in E_n$ such that $\| x_n \| \geqq \alpha_n / 2$. Once the x_n's are constructed, set

$$K_n = \Big\{ \sum_{i=1}^{n} \varepsilon_i x_i : (\varepsilon_i) \in \{ -1, +1 \}^n \Big\} \quad \text{and} \quad K = \bigcup_{n \in \mathrm{N}} K_n .$$

The contradiction is then obtained in the same way as in the proof of Theorem 3.1, distinguishing the cases K bounded and K unbounded.

THEOREM 5.2. *Let X be a C^p-smooth Banach space of exact cotype p (p even integer $\geqq 2$). Then X contains an isomorphic copy of $l^p(\mathrm{N})$.*

REMARK. According to Proposition 5.2 and Lemma 2.1, if p is an odd integer ($p \geqq 2$) and if X is of exact cotype p, then X cannot be C^p-smooth. Therefore, the hypothesis of Theorem 5.2 never holds for p an odd integer, $p \geqq 2$.

COROLLARY 5.3. *Let X be a C^∞-smooth Banach space. Then, at least one of the following statements is satisfied:*

(1) *X contains an isomorphic copy of $c_0(\mathrm{N})$.*

(2) *There is an even integer $p \geqq 2$ such that X contains an isomorphic copy of $l^p(\mathrm{N})$.*

PROOF OF COROLLARY 5.3. Let X be a C^∞-smooth Banach space. If X does not contain an isomorphic copy of $c_0(\mathrm{N})$, then, by Theorem 3.1, there exists p an even integer, $p \geqq 2$, such that X is of exact cotype p. The corollary follows then from Theorem 5.2.

PROOF OF THEOREM 5.2. Under the hypothesis of Theorem 5.2, we can apply Proposition 5.1. There is an infinite dimensional subspace Z of X and a p-linear continuous symmetric form Φ on X such that, for every $x \in Z$, $\Phi(x, x, \ldots, x) \geqq \| x \|^p$. Moreover, none of the infinite dimensional subspaces of Z is of cotype $p - 1$.

Let (x_n) be a basic sequence of norm 1 elements of Z. We notice that Proposition 2.4 has the following consequence: suppose that for $1 \leqq i \leqq m$, ψ_i

is a k_i-linear continuous symmetric form with $k_i \leqq p-1$. Then there is a block basic sequence (y_k) of (x_n) satisfying $\| y_k \| = 1$ and

$$\psi_i(y_k, y_k, \ldots, y_k) \leqq \delta_k \quad \text{for every } k \in \mathbf{N} \text{ and every } i \in \{1, 2, \ldots, m\}$$

where (δ_k) is a sequence of real numbers decreasing to 0 and satisfying

$$\sum_{\substack{1 \leqq k_1 \leqq k_2 \leqq \cdots \leqq k_p < +\infty \\ k_1 \neq k_p}} \delta_{k_p} \leqq p/(2 \cdot p!).$$

This remark allows us to construct a block basic sequence (y_k) of (x_n), $\| y_k \| = 1$, such that for every sequence of integers $1 \leqq k_1 \leqq k_2 \leqq \cdots \leqq k_p < +\infty$

$$\Phi(y_{k_1}, y_{k_2}, \ldots, y_{k_p}) \geqq 1 \qquad \text{if } k_1 = k_2 = \cdots = k_p,$$

$$\Phi(y_{k_1}, y_{k_2}, \ldots, y_{k_p}) \leqq \delta_{k_p} \qquad \text{otherwise.}$$

Let (e_k) be the unit vector basis of $l^p(\mathbf{N})$ and define

$$\mathrm{F}\colon \quad l^p(\mathbf{N}) \to X,$$

$$\sum_{k=1}^{+\infty} a_k e_k \to \sum_{k=1}^{+\infty} a_k y_k.$$

We claim that F is an isomorphism from $l^p(\mathbf{N})$ into X. To show this, it is enough to find two constants $A, B > 0$ such that, for every $y \in l^p(\mathbf{N})$ with finite support,

$$A \| y \|_{l^p(\mathbf{N})} \leqq \| F(y) \|_X \leqq B \| y \|_{l^p(\mathbf{N})}.$$

Let us denote, for $x \in Z$, $\varphi(x) = \Phi(x, x, \ldots, x)$ and $C = \sup\{ |\varphi(x)| : x \in Z, \| x \| = 1\}$. Let $y = \Sigma_{k=1}^n a_k e_k$. We have $F(y) = \Sigma_{k=1}^n a_k y_k$ and

$$\begin{aligned} &\varphi(F(y)) \\ &= \varphi\left(\sum_{k=1}^n a_k y_k\right) \\ &\geqq \sum_{k=1}^n a_k^p - \sum_{\substack{1 \leqq k_1 \leqq \cdots \leqq k_p \leqq n \\ k_1 \neq k_p}} K(k_1, k_2, \ldots, k_p) | a_{k_1} a_{k_2} \cdots a_{k_p} \Phi(y_{k_1}, y_{k_2}, \ldots, y_{k_p}) | \\ &\geqq \sum_{k=1}^n a_k^p - p! \sum_{\substack{1 \leqq k_1 \leqq \cdots \leqq k_p \leqq n \\ k_1 \neq k_p}} | a_{k_1} a_{k_2} \cdots a_{k_p} | \delta_{k_p} \end{aligned}$$

$$\geqq \sum_{k=1}^{n} a_k^p - p!/p \sum_{\substack{1 \leqq k_1 \leqq \cdots \leqq k_p \leqq n \\ k_1 \neq k_p}} (a_{k_1}^p + a_{k_2}^p + \cdots + a_{k_p}^p)\delta_{k_p}$$

$$\geqq \sum_{k=1}^{n} a_k^p - \left(\sum_{k=1}^{n} a_k^p\right)\left(p!/p \sum_{\substack{1 \leqq k_1 \leqq \cdots \leqq k_p \leqq n \\ k_1 \neq k_p}} \delta_{k_p}\right)$$

$$\geqq \frac{1}{2}\left(\sum_{k=1}^{n} a_k^p\right).$$

Therefore

$$\| y \|_{l^p(\mathbf{N})}^p = \sum_{k=1}^{n} a_k^p \leqq 2\varphi(F(y)) \leqq 2C \| F(y) \|^p.$$

This yields the left-hand-side inequality. On the other hand, a similar computation gives

$$\| F(y) \|^p \leqq \varphi(F(y)) \leqq (C + \tfrac{1}{2})\left(\sum_{k=1}^{n} a_k^p\right) = (C + \tfrac{1}{2}) \| y \|_p$$

and this proves Theorem 5.2.

REMARKS. (1) It is known that $c_0(\mathbf{N})$ and $l^p(\mathbf{N})$, for p an even integer, have equivalent norms which are C^∞-smooth away from the origin and therefore these spaces admit a C^∞-smooth bump function. Corollary 5.3 states that, conversely, a Banach space admitting a C^∞-smooth bump function contains a subspace isomorphic to one of these elementary examples ($c_0(\mathbf{N})$ and $l^p(\mathbf{N})$ for p an even integer).

(2) It is an easy exercise to show that the Banach space

$$X = (l^2(\mathbf{N}) \oplus l^4(\mathbf{N}) \oplus \cdots \oplus l^{2k}(\mathbf{N}) \oplus \cdots)_0$$

of sequences $(x_1, x_2, \ldots, x_k, \ldots) \in l^2(\mathbf{N}) \times l^4(\mathbf{N}) \times \cdots \times l^{2k}(\mathbf{N}) \times \cdots$ such that $\lim_{k \to +\infty} \| x_k \|_{l^{2k}(\mathbf{N})} = 0$ endowed with the norm

$$\| (x_1, x_2, \ldots, x_k, \ldots) \| = \sup_{k \geqq 1} \| x_k \|_{l^{2k}(\mathbf{N})}$$

is a C^∞-smooth Banach space (the proof is a refinement of the proof that $c_0(\mathbf{N})$ is C^∞-smooth appearing in [19]) and X contains all the elementary examples of C^∞-smooth Banach spaces, i.e. $c_0(\mathbf{N})$ and $l^p(\mathbf{N})$ for every p an even integer $\geqq 2$.

(3) The results of this paper show that there are very few Banach spaces

admitting a C^∞-smooth bump function. Another result in the same direction was obtained recently in [14], where it is shown that the only Orlicz sequence spaces admitting a C^∞-smooth bump function are the spaces $l^p(\mathbf{N})$ where p is an even integer $\geqq 2$.

(4) In view of remark (3), one can ask if the only Banach spaces admitting a C^∞-smooth bump function are the subspaces of the spaces $L^p(\Omega)$ where p is an even integer $\geqq 2$. This is not the case since one can easily check, just looking at the formula of the norm, that the Banach space $(\bigoplus l^4(\mathbf{N}))_8$ of all sequences of scalars $(a_{n,k})$ such that $\| (a_{n,k}) \| = (\Sigma_{k=1}^{+\infty}(\Sigma_{n=1}^{+\infty}(a_{n,k})^4)^2)^{1/8}$ is C^∞-smooth (the 8th power of the norm is an analytic function on $(\bigoplus l^4(\mathbf{N}))_8$). On the other hand, $X = (\bigoplus l^4(\mathbf{N}))_8$ cannot be isomorphic to a subspace of some $L^p(\Omega)$ because if X is a subspace of $L^p(\Omega)$, then X can contain $l^2(\mathbf{N})$ or $l^p(\mathbf{N})$, but cannot contain $l^q(\mathbf{N})$ for $q \neq 2$ and $q \neq p$, which contradicts the fact that X contains both $l^4(\mathbf{N})$ and $l^8(\mathbf{N})$.

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REFERENCES

1. C. Bessaga and A. Pelczynski, *On basis and unconditional convergence of series in a Banach space*, Studia Math. **17** (1958), 151–164.

2. R. Bonic and J. Frampton, *Smooth functions on Banach manifolds*, J. Math. Mech. **15** (1966), 877–898.

3. R. Bourgin, *Geometrical aspects of sets with the Radon Nikodym property*, Lecture Notes in Math. **993**, Springer-Verlag, Berlin, 1983.

4. K. Deimling, *Non-Linear Functional Analysis*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.

5. R. Deville and V. Zizler, *A note on cotype of smooth spaces*, to appear.

6. M. Fabian, *Lipschitz smooth points of convex functions and isomorphic characterization of Hilbert spaces*, Proc. London Math. Soc. (3) **51** (1985), 113–126.

7. M. Fabian, J.H.M. Whitfield and V. Zizler, *Norms with locally Lipschitzian derivatives*, Isr. J. Math. **44** (1983), 262–276.

8. M. Fabian, D. Preiss, J. H. Whitfield and V. Zizler, *Separating polynomials in Banach spaces*, to appear.

9. G. Godefroy and V. E. Zizler, forthcoming book on renormings.

10. J. Hoffmann-Jørgensen, *Probability and geometry of Banach spaces*, Lecture Notes in Math. **598**, Springer-Verlag, Berlin, 1976.

11. J. Kurzweil, *On approximation in real Banach spaces*, Studia Math. **14** (1954), 213–231.

12. S. Kwapien, *Isomorphic characterization of inner product spaces by orthogonal series with vector valued coefficients*, Studia Math. **44** (1972), 583–595.

13. E. B. Leach and J. H. M. Whitfield, *Differentiable functions and rough norms on Banach spaces*, Proc. Am. Math. Soc. **33** (1972), 120–126.

14. R. Maleev and S. Troyanski, *Smooth functions in Orlicz spaces*, Proceedings of Workshop on Banach Spaces, Iowa 87, U.S.A., Contemporary Mathematics **85**, Am. Math. Soc., pp. 355–370.

15. B. M. Markarov, *One characteristic of Hilbert space*, Mat. Zametri **26** (1979), 739–746.

16. B. M. Makarov, *A condition for the isomorphism of a Banach space having the Orlicz property to a Hilbert*, Zap. Nauchn. Semin. Leningr. **126** (1983), 138–142.

17. B. Maurey and G. Pisier, *Séries de variables aléatoires vectoriélles indépéndantes et propriétés géometriques des espaces de Banach*, Studia Math. **48** (1976), 45–90.

18. V. Z. Meshkov, *Smoothness properties in Banach spaces*, Studia Math. **63** (1978), 111–123.

19. J. Pechanec, J. H. M. Whitfield and V. Zizler, *Norms locally dependent on finitely many coordinates*, Ann. Acad. Brasil **53** (1981), 415–417.

20. G. Pisier, *Factorization of linear operators and geometry of Banach spaces*, C.B.M.S. Regional Conference Series in Math. 60, Am. Math. Soc.

21. D. Preiss, *Differentiability of Lipschitz functions on Banach spaces*, to appear.